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Economic mapping to the renormalization group scaling of stock markets

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Abstract. We make an attempt to map a simple economically motivated model for price evolution [J. Phys. A **33**, 3637 (2000)] to the phenomenological renormalization group scaling of stock markets. This mapping gives insight into the critical exponents and the renormalization group predictions for the log-periodic oscillations preceding some stock market crashes from the perspective of non-linear changes in *'the level of stock'*.

PACS. 89.90+n Other topics in aeras of applied and interdisciplinary physics – 64.60Ak Renormalization-group, fractal, and percolation studies of phase transitions

1 Introduction

Several papers have appeared in recent years showing increasing evidence that at least some market crashes are anticipated by a power law behaviour of the stock market index which fluctuates with oscillations that are periodic in the logarithm of the time to crash (see, for example, [1–6]). From these observations, it has been argued that there is a close relation between stock market crashes and renormalization group (RG) theory [1]. Precursory logarithm- (log-)periodic patterns can also emerge from percolation models by applying the cluster concept to groups of investors acting collectively [7,8].

The RG approach has been found to model stock market time evolution remarkably well, predicting the existence of large price crashes. However a possible universality for the real exponents quantifying the observed behaviour in market prices, which would define a crash, has not yet been established [2]. Unlike in the case of systems in thermodynamic equilibrium, there is no known underlying Hamiltonian from which RG critical exponents could be deduced. In this paper we propose a simplified dynamics for the price evolution and make an attempt to map this dynamics to the RG predictions. We show how the simplest, non-linear economic model proposed by the author [9] may be mapped onto the non-linear RG scaling of stock markets in order to understand the critical exponents in terms of relevant economic variables such as the demand and supply of a commodity (or product). As an illustrative example, we apply the mapping to the NY Standard & Poor (S&P)500 index crash of October 1987.

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2 Non-linear renormalization group generalization

In analogy with RG theory, it is assumed that the temporal variation of the stock market index $I(t)$ is related to future events at t' by the transformations [1,2]

$$x' = \phi(x), \tag{1}$$

$$F(x) \equiv I(t_c) - I(t) = g(x) + \frac{1}{\mu} F\Big(\phi(x)\Big), \tag{2}$$

where $x = t_c - t$. ϕ is called the RG flow map and μ is a constant describing the scaling of I on the rescaling of t in equation (1). $F = 0$ at the critical point t_c (the time of a large crash), $g(x)$ represents the non-singular part of the function $F(x)$, which is assumed to be continuous, and $\phi(x)$ is assumed to be differentiable.

An extension of these results to a more general RG approach begins by considering that the solution of the RG equation (2), in conjunction with equation (1) and the linear approximation $\phi(x) = \hat{\lambda} x$, can be rewritten as

$$\frac{\mathrm{d}F(x)}{\mathrm{d}\log x} = \alpha F(x). \tag{3}$$

This defines limiting power laws as $t \to t_c$. Equation (3) is then extended to include corrections to the power law with log-periodicity by introducing the amplitude B and phase ψ of $F(x) = B\mathrm{e}^{\mathrm{i}\psi(x)}$. The symmetry law used is the phase shift that should keep the observable constant under a change of units [2]. This leads us to postulate the following Landau expansion [10]:

$$\frac{\mathrm{d}F(x)}{\mathrm{d}\log x} = (\alpha + \mathrm{i}\omega)F(x) + (\eta + \mathrm{i}\kappa)|F(x)|^2 F(x) + \mathcal{O}(F^5). \tag{4}$$

where $\alpha > 0$, ω, η and κ are real coefficients and $\mathcal{O}(F^5)$ represents higher order terms which will be neglected.

In terms of the amplitude and phase of F, equation (4) yields

$$\frac{\partial B}{\partial \log x} = \alpha B + \eta B^3 + \cdots \quad ; \quad \frac{\partial \psi}{\partial \log x} = \omega + \kappa B^2 + \cdots \tag{5}$$

with solutions

$$B^2 = \frac{(\frac{x}{x_0})^{2\alpha}}{1 + (\frac{x}{x_0})^{2\alpha}} \, B_\infty^2, \tag{6}$$

and

$$\psi = \omega \log(\frac{x}{x_0}) + \frac{\Delta\omega}{2\alpha} \log\left(1 + (\frac{x}{x_0})^{2\alpha}\right), \tag{7}$$

where $B_\infty^2 = \alpha/|\eta|$, $\Delta\omega = B_\infty^2 \kappa$ and x_0 is an arbitrary coefficient characterizing the time scale. These general forms lead to the following solutions of the non-linear RG equation (4):

$$I(\tau) = A_1 + A_2 \frac{(\tau_c - \tau)^\alpha}{\sqrt{1 + (\frac{\tau_c - \tau}{\Delta t})^{2\alpha}}} \times \left[1 + A_3 \cos\left(\omega \log(\tau_c - \tau) + \frac{\Delta\omega}{2\alpha} \log\left(1 + \left(\frac{\tau_c - \tau}{\Delta t}\right)^{2\alpha}\right)\right)\right], \tag{8}$$

where $\tau = t/\phi$, $\Delta t = x_0$ and $A_{i=1,2,3}$ are linear variables.

3 Non-linear economic model

Within our economic model [9], only one stock of the commodity is assumed and the market is considered competitive, so it self-organizes to determine the behaviour of the asset price p. We derive a dynamic price equation which results from the prevailing market conditions in terms of the excess demand function $E(p) = D(p) - Q(p)$, where D and Q are the demand and supply functions, respectively. In our description, an asterisks (*) denotes quantities in equilibrium. All variables are dimensionless.

In a competitive market the rate of price increase usually is a functional of $E(p)$ such that $\mathrm{d}p/\mathrm{d}t \equiv f[E(p)]$ [11]. Considering that in general a commodity can be stored, then stocks of the commodity build up when the flow of output exceeds the flow of demand and *vice versa*. The rate at which *'the level of stock'* S changes can be approximated as $\mathrm{d}S/\mathrm{d}t = Q(p) - D(p)$. Thus a price adjustment relation that takes into account deviations of the stock level S above certain optimal level S_o (to meet any demand reasonably quickly) is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma \frac{\mathrm{d}S}{\mathrm{d}t} + \lambda(S_o - S), \tag{9}$$

where γ (*i.e.*, the inverse of excess demand required to move prices by one unity [12]) and λ are positive factors. For $\lambda > 0$, prices increase when stock levels are low and rise when they are high (with respect to S_o). When $\lambda = 0$, the price adjusts at a rate proportional to the rate at which stocks are either rising or falling.

For all asset prices $p(t)$, non-linear forms for the quantities D demanded and Q supplied, are postulated such that

$$D(p) = d^* + d_o[\, 1 - \frac{\delta^2}{2!}(p - p^*)^2 + \ldots](p - p^*),$$
$$Q(p) = q^* + q_o[\, 1 - \frac{\delta^2}{2!}(p - p^*)^2 + \ldots](p - p^*), \tag{10}$$

where d_o, q_o and $d^* = D(p^*)$, $q^* = Q(p^*)$ are arbitrary coefficients (related to material costs, wage rate, etc.), $p^* = p(t^*)$ is an equilibrium price and $\delta < 0$ is an order parameter as discussed in [9]. Expansion terms $\mathcal{O}(5)$ are here neglected.

Considering the simplest, complete economic model as in [11], we assume that S_o depends linearly on the demand; *e.g.*, $S_o = \ell_o + \ell D$, with ℓ_o a constant and ℓ satisfying

$$\ell \equiv \frac{\gamma\beta_o}{\lambda d_o} \quad , \tag{11}$$

where $\beta_o \equiv q_o - d_o$. We have shown that only this condition can lead to solutions of the dynamic price equation in real space [9]. Therefore, in equilibrium (where $\frac{\mathrm{d}p}{\mathrm{d}t}|_{p^*} = 0$ and $\frac{\mathrm{d}S}{\mathrm{d}t}|_{S^*} = 0$, so that demand equals supply and $S = S_o$), we obtain $d^* - q^* = 0$ and $S^* = \ell_o + \ell(d^* + d_o p^*)$.

After some algebra, the second derivative of the price adjustment equation (9) of one commodity can be approximated as

$$\frac{\mathrm{d}^2 p}{\mathrm{d}t^2} \;\approx\; -\lambda\beta_o(p - p^*) + \frac{\delta^2 \lambda\beta_o}{2}(p - p^*)^3. \tag{12}$$

For $\delta \neq 0$ and $[\lambda\beta_o, \delta^2\lambda\beta_o/2] > 0$, it has the well-known kink solutions

$$p(t) = p^* + \frac{\sqrt{2}}{\delta} \; \tanh\left(\sqrt{\frac{\lambda\beta_o}{2}}\,(t - t^*)\right), \tag{13}$$

such that β_o is positive. As in a competitive market economy the demand for a commodity falls when its price increases, then it is reasonable to assume $d_o < 0$ in equation (10). And as the price rises, the supply usually also increases; hence in general one also assumes $q_o > 0$. These conditions yield $\beta_o > 0$ as required and also $d_o\ell > 0$.

4 The mapping

We show next how a mapping can be established in order to identify the real, phenomenological RG critical exponents α and η in terms of our non-linear economic model variables. From a comparison between equations (5) and

(12) we identify the following relation between the expansion terms

$$\alpha B \rightleftharpoons -\,\delta\beta_{\rm o}(p-p^*). \tag{14}$$

From this simple mapping we make an attempt to understand RG modelling of stock markets and use it to analyse and predict financial crashes in analogy to critical points studied in physics with log-periodic correction to scaling [2].

If for $t \to t^*$ we approximate $\sinh\left(\sqrt{\frac{\lambda\beta_{\rm o}}{2}}\,(t-t^*)\right) \approx \sqrt{\frac{\lambda\beta_{\rm o}}{2}}\,(t-t^*)$ such that

$$\tanh\left(\sqrt{\frac{\lambda\beta_{\rm o}}{2}}\,(t-t^*)\right) \approx \frac{\sqrt{\frac{\lambda\beta_{\rm o}}{2}}\,(t-t^*)}{\sqrt{1+\frac{\lambda\beta_{\rm o}}{2}\,(t-t^*)^2}}\,\cdot \tag{15}$$

Then using the mapping in equation (14) and the solutions for B and p given by equations (6) and (13), respectively, we obtain

$$\alpha \rightleftharpoons \lim_{t\to t_{\rm c}\rightleftharpoons t^*} \frac{\log\left((t^*-t)/\sqrt{2/\lambda\beta_{\rm o}}\right)}{\log(x/x_{\rm o})} \;\to\; 1,$$
$$|\eta| \rightleftharpoons \frac{\delta^2}{2\lambda^2\beta_{\rm o}^2}, \tag{16}$$

such that, as before $x = t_{\rm c} - t$ and $x_{\rm o} = \Delta t$, and $\Delta t \to \sqrt{2/\lambda\beta_{\rm o}}$. The above α is consistent with the definition of critical exponents [10].

Using these mappings for $\alpha \to 1$ and η, it is straightforward to show that they also relate the second series expansion terms between equations (5) and (12), namely: $\eta B^3 \rightleftharpoons \frac{\delta^2\lambda\beta_{\rm o}}{2}(p-p^*)^3$ provided that $\delta < 0$. Hence, in terms of our non-linear economic model variables, we find the following extended solutions in analogy with the non-linear RG framework:

$$I\,(t) = A_1 + A_2\frac{(t^*-t)}{\sqrt{1+\left(\frac{t^*-t}{\Delta t}\right)^2}}\left[1 + A_3\cos\left(\omega\log\,(t^*-t)\right.\right.$$
$$\left.\left. +\frac{\Delta\omega}{2}\log\left(1+\left(\frac{t^*-t}{\Delta t}\right)^2\right)\right)\right], \tag{17}$$

with $\tau \rightleftharpoons t$, $\tau_{\rm c} \rightleftharpoons t^*$, $\Delta\omega \rightleftharpoons 2(\lambda\beta_{\rm o}/\delta)^2\kappa$, and $\Delta\tau \rightleftharpoons \sqrt{2/\lambda\beta_{\rm o}}$.

5 Discussion

As an example, we apply the present mapping to the S&P500 index. In Figure 1 we show the fit of equation (17) to the time dependence of the logarithm of the NY S&P 500 index from January 1985 to the October 1987 crash. The parameters used in this illustrative curve (full line) are: $A_1 = 5.79$, $A_2 = -0.32$, $A_3 = 0.059$, $w = 6.47$, $\Delta t = 2.29$, $\Delta\omega = 15.42$ and $t^* = 87.70$ decimal years (with $rms = 0.02$). The parameter values used for the best fit of equation (8) (dotted lines in the figure) are those found in [2].

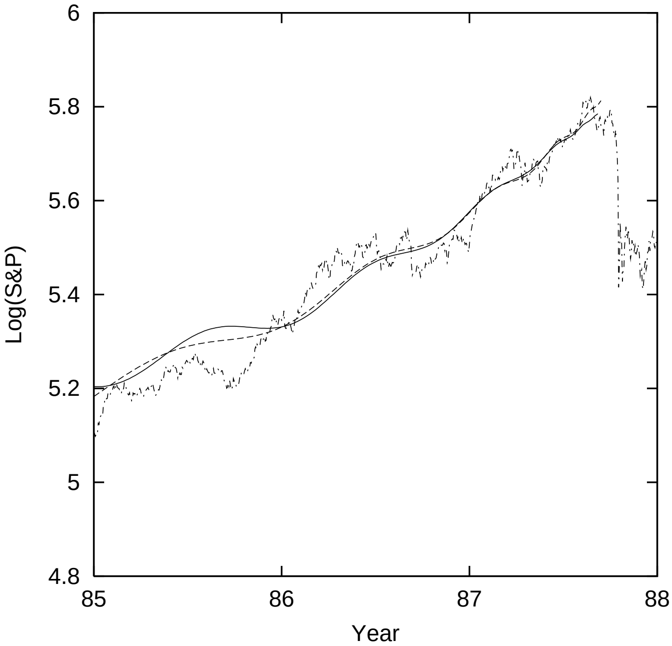

Fig. 1. Time dependence of the logarithm of the NY S&P 500 index from January 1985 to the October 1987 crash. The full line curve is the fit of equation (17) for a year scale of 2.29 years and the dotted lines curves the fit of equation (8) with $\Delta t = 11$ years.

Similarly to the non-linear RG scaling results, we see that the general trend of the S&P 500 data is also reproduced by the mapping of our economic model in the limit $t \to t^*$ so that $\alpha \to 1$ as deduced from equation (16). This is to be expected due to the oscillations that are periodic in the logarithm of the time to crash appearing in both equations (8) and (17).

The modulation of the logarithm frequency does not changes significatively when one approaches the critical point $t_{\rm c} \rightleftharpoons t^*$. On relatively short time scales, spikes are not accounted for in the two equations due to complex self-organizing phenomena in stock markets other than the one analysed here. The log-periodic structure found prior to crashes implies the existence of a hierarchy of time scales [2]. The choice of the parameters $A_{i=1,2,3}$ is empirical in both cases.

Our mapping may give insight into the nature of market crashes from the new perspective of the demand D and supply Q of a commodity. Our non-linear expressions for D and Q of equation (10) are plotted in Figure 2 and are justified as follows: when $|\delta p| \ll 1$, these functions display similar behaviour to the (commonly used) linear p-dependence for D and Q. Even more importantly, they depict the fact that as price falls, the demand for a commodity can increase in agreement with one of the basic principles of economy. On the other hand, our choice for

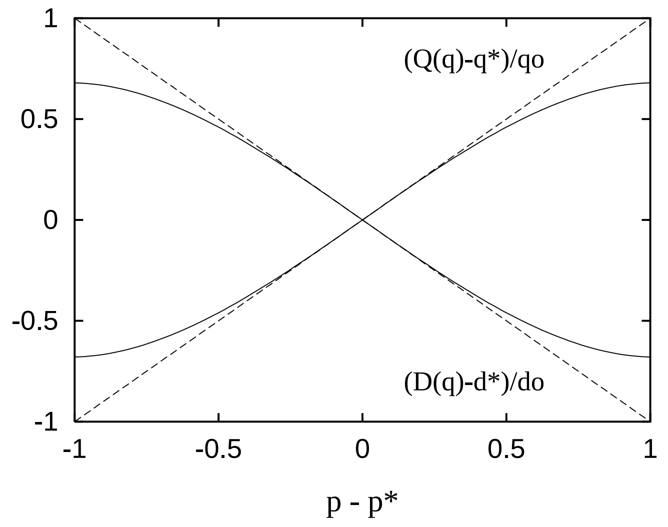

Fig. 2. Non-linear forms for the demand and supply functions of equation (10) with $q_o > 0$, $d_o < 0$, $|\delta| = 0.8$ (full lines) and $\delta = 0$ (dotted lines).

Q (with $q_o > 0$) also follows the typical behaviour observed in a competitive market (where no individual producer can set his own desired price). That is, the higher the price, the higher the profit, then the higher the supply (see [9] for a more extensive discussion).

We have related the critical exponents α and η to the relevant variables of our non-linear economic model based on observed laws for D and Q [9]. As $t \to t^*$ we have identified $\alpha \to 1$ whereas the η exponent given in equation (16) is found to depend on the order parameter $\delta < 0$, relating D and Q of a commodity as in equation (10) in conjunction with the economic model factors λ and β_o under the limiting constraints of equation (15).

Since our Δt coefficient also depends on λ and β_o, these factors drive the observed effects for $t/t_c \gg 1$, where there is a saturation of the function $I(t)$, and for $t \to t_c$, where the log-frequency shifts from $\frac{\omega+\Delta\omega}{2\pi}$ to $\frac{\omega}{2\pi}$. In economic terms, these features are directly related to the temporal adjustments of *'the level of stock'* S as given in equation (9).

The concept of a certain optimal level of stock is well-known in economic theory for stocks [11]. Planning ahead to have suitable *'level of stock'* is essential. If production had to be stopped every time a company ran out of raw materials, the time wasted would cost a fortune.

Indeed stock is held for a variety of reasons. There may be stocks of raw materials ready for production, stocks of work-in-progress (*e.g.*, production parts) or stocks of finished goods. Whichever they are it is vital for a company to control *'the level of stock'* very carefully. Too little and they may run into production problems, but too much and they have tied up money unnecessarily. Low *'level of stock'* – say 10% – would certainly be adequate if production levels could be maintained during the years. Usually *'the level of stock'* needs to be adjusted as the marketing year progresses. Stock is considered to a current asset because it can be converted into cash reasonably quickly. On the other hand, producers can also carry some stock surplus as a way to speculate on prices.

6 Conclusions

We have shown that an economic mapping to the RG scaling of stock markets reproduces reasonably well the trends in the S&P 500 index in the vicinity of the time of crash. As in the RG model, the mapping predicts the existence of a crash due to corrections to the power law with log-periodicity but such that $\alpha \to 1$. The main point of this work follows that of the RG model. That is, the underlying cause of the crash must be searched years in advance by looking at the progressive accelerating ascent of the market price. Our formalism differs in that we have a shorter year scale of about $\Delta t = 2.29$ years compared to the fitting of $\Delta t = 11$ years reported in [2] in which these cooperative phenomena are progressively being constructed. In this period of time, one should also look for the appearance of non-linearities in the behaviour of the demand and supply functions (or, alternatively, in *'the level of stock'*) of the commodities prior to the crash as shown here.

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